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Strongly Principal Ideals of Rings with Involution

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Abstract

The notion of strongly principal ideal groups for associative rings was introduced in [3] and its several properties were studied in [2] by using cyclic groups. Motivated by these concepts, we introduce here *-cyclic groups and strongly principal *-ideal ring groups for rings with involution and investigate their structural properties by attaching involution on their corresponding ground groups.

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1. Introduction

Let P be a ring property. A group G is said to be $P-$group, in the sense of Feigelstock and Schlussel [3] (see also [2]), if there exists a $P-$ring R such that $G = R^+$. A group G is said to be nil, again in the sense of the above cited references, in case the only ring R with $G = R^+$ is the zero ring. If G is not nil and every ring R, with $R^2 \neq 0$ and $G = R$ is a $P-$ring, then G is said to be strongly $P-group$. By using these concepts, strongly principal ideals are thoroughly investigated. For rings with involution, we introduce here the notion of strongly principal *-ideal ring groups and study their structural properties. To achieve this goal and to make the calculations simpler we have introduced *-cyclic groups, which is generated by an element a and its involutary image a^* in the group. Moreover, we study some structural properties of *-cyclic groups as well. In particular a formula (Corollary 3.2) for computing the number of involutions for abelian groups is obtained. Their direct sum and direct summand propert ies are also outlined. We have used *-cyclic groups to obtain various properties and classifications of (strongly) principal *-ideal ring groups. In particular, it is noticed in Theorem 4.7 that there exists no mixed strongly principal *-ideal ring groups.

Throughout we assume that all groups are additive abelian and all rings are associative. If R is a ring, then its underlying additive group is denoted by $G = R^+$. If $x \in R$, then $\langle x \rangle$ (respectively (x)) means the ideal of R (respectively the subgroup of G) generated by x.

A ring R (respectively a group G) together with a unary operation $*$ is said to be a *ring* (respectively, *a group*) *with involution* in case for all $a, b \in R$,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad \textit{and} \quad (ab)^* = b^* a^*$$

(respectively, for all $a, b \in G$, $(a^*)^* = a$, *and* $(a+b)^* = a^* + b^*$).Thus the involution on R is an antiisomorphism of order two.

For commutative rings, the identity mapping is clearly an involution. Nevertheless, every group has at least one involution, namely, the unary operation of taking inverse; that is $g^* = -g$ for every $g \in G$.

Let a group G be decomposed into its subgroups as $G = H \oplus K$. If G has an involution $*$, then $*$ is said to be *changeless involution* in case $g^* = (h^*, k^*)$, $\forall\ g = (h, k) \in H \oplus K$ (see[1]).

A group G is said to be **-cyclic* if for some $a \in G$, $G = (a) + (a^*)$, which indeed one may rewrite as $G = (a)^* = (a, a^*)$. Clearly, every cyclic group is *-cyclic, but the converse is not true in general (see an example in Section 3).

A nonzero ideal I of an involution ring R (a nonzero subgroup H of an involution group G) which is closed under involution is termed as a **-ideal* $(I \lhd^* R)$ (*respectively a *-subgroup* $(H \leq^* G)$); that is

$$I^* = \{a^* \in R \mid a \in I\} \subseteq I \quad .$$

A subring A of R is said to be a *biideal* of R if $ARA \subseteq A$ and a **-biideal* if, in addition, it is closed under the involution of R. A is called a *principal *-biideal* (see [7]), if

$$A = \langle a \rangle^*_{bi} = \mathbb{Z}a + \mathbb{Z}a^* + aRa + a^*Ra + aRa^* + a^*Ra^*.$$

On the same ground a *principal *-ideal* is defined. A *principal *-ideal* I is a *-ideal generated by a single element. This means that, for some $a \in R$, one may write:

$$I = \langle a \rangle^* = \mathbb{Z}a + \mathbb{Z}a^* + aR + Ra + RaR + a^*R + Ra^* + Ra^*R.$$

Thus, it can easily be deduced that

$$I = \langle a\rangle + \langle a^*\rangle = \langle a, a^*\rangle \,.$$

A ring with involution $*$ is said to be *principal *-ideal ring* if each *-ideal is a principal *-ideal.

A group$G$ is called *strongly principal *-ideal ring group*, if G is not nil and every ring R with involution satisfying $R^2 \neq 0$ and $G = R^+$, is a principal *-ideal ring.

Let $f : A \longrightarrow B$ be a group or a ring homomorphism. If A and B are equipped with some involutions $*_A$ and $*_B$ such that $f(a^{*_A}) = [f(a)]^{*_B}$, then we say that f is an *involution preserved homomorphism.* If f is an *involution preserved isomorphism,* then we will write $A \overset{*}{\cong} B$. It is clear that *-subgroups and *-ideals are preserved under such isomorphisms. Moreover, if $A \cong B$, as a group or a ring, then every involution on A induced an involution on B.

In sections 2 and 3,we give some elementary properties for *-cyclic groups. Furthermore ,in section 4,(strongly) principal *-ideal ring groups are widely studied.

2. Some Elementary Properties

Lemma 2.1. *Let G* be *a group with involution $*$. Then the following subgroups of G are closed under the involution $*$.*

(a) *nG, $\forall n \in \mathbb{Z}$.*

(b) *The torsion subgroup G_t of G.*

(c) *For any prime p, every p-primary subgroup G_p of G.*

(d) *The maximal divisible subgroup of G.*

(e) *The subgroup $G[m] = \{g \in G \mid mg = 0\}$ of G,* for some integer m.

Proof : (a) Let $x \in nG$. Then $x = ng$ for some $g \in G$, hence $x^* = ng^*$ and $g^* \in G$. So $x^* \in nG$ and nG is closed under involution.

(b) Let $x \in G_t$. Then there exists a positive integer n such that $nx = 0$. Hence $nx^* = 0$ and $x^* \in G$ follows..

(c) Let $x \in G_p$. Then $|x| = p^n$ for some positive integer n and $p^n x = 0$, implies $p^n x^* = 0$. Hence $x^* \in G_p$.

(d) Let D be the maximal divisible subgroup of G. If $x \in D$, then there exists $y \in D$ such that $x = ny$ for any positive integer n ,whence $x^* = ny^*$. Since D is the maximal divisible subgroup, $x^*, y^* \in D$, therefore D is closed under involution.

(e) Let $x \in G[m]$, then $mx = 0$,whence $mx^* = 0$ and $x^* \in G[m]$ follows.■

Corollary 2.2. *In every involution ring R, nR, $R[n]$, R_t, R_p and the maximal divisible ideal D are *-ideals.*

Proof : It is clear that $G = R^+$ has involution; the same involution of R. So from Lemma 2.1, $nR,\ R[n],\ R_t,\ R_p$ and the maximal divisible subgroup D are *-subgroups of G. Since $nR,\ R[n],\ R_t,\ R_p$,and D are ideals in R (see [5]), hence $nR,\ R[n],\ R_t,\ R_p$ and D are *-ideals in every involution ring R. ■

Lemma 2.3. (a) *Every direct sum of involution groups is an involution group.*

(b) *Every direct summand of a group with a changeless involution is an involution group.*

(c) *If a direct summand of a group has an involution, then the group has an involution.*

Proof : (a) Let $G = H \oplus K$, where H and K are groups with involutions $*_H$ and $*_K$, respectively. Then for every $g = (h, k) \in G$, where $h \in H$ and $k \in K$,define the involution $*_G$ on G by

$$g^{*_G} = (h^{*_H}, k^{*_K}).$$

Because of the unique representation of each element, $*_G$ becomes a unary operation on G. Further,

$$(g^{*_G})^{*_G} = ((h^{*_H})^{*_H}, (k^{*_K})^{*_K})) = (h, k) = g.$$

Assume that $g_i \in G$, with $g_i = (h_i, k_i)$, where $h_i \in H$ and $k_i \in K$. Then

$$\begin{aligned}(g_1 + g_2)^{*_G} &= ((h_1 + h_2)^{*_H}, (k_1 + k_2)^{*_K}) = ((h_1^{*_H} + h_2^{*_H}), (k_1^{*_K} + k_2^{*_K}))\\ &= (h_1^{*_H}, k_1^{*_K}) + (h_2^{*_H}, k_2^{*_K}) = g_1^{*_G} + g_2^{*_G}.\end{aligned}$$

Hence $*_G$ is an involution on G ;it is in fact the changeless involution on G .

The proof can analogously be extended to finite as well as to arbitrary direct sums.

(b) Let $G = H \oplus K$. Set $H' = H \oplus 0$ and $K' = 0 \oplus K$. Clearly, H' and K' are direct summands and subgroups of G. Assume that $*$ is the changeless involution on G. Then $* |_{H'}$(involution on G restricted to H') is an involution on H' and $* |_{K'}$is an involution on K'. Also, $H \overset{*}{\cong} H'$ and $K \overset{*}{\cong} K'$. Hence (b) is proved.

(c) Let $G = H \oplus K$ and H be a group with an involution $*_H$. Then for every $g = (h, k) \in G$, where $h \in H$ and $k \in K$, define an operation $*_G$ on G by

$$g^{*_G} = (h^{*_H}, k).$$

Clearly, $*_G$ is the changeless involution on G.■

Parts (a) and (b) of Lemma 2.3 can easily be extended to rings, subrings and ideals. But for part (c) we need the following modification.

Corollary 2.4. *Let $R = A \oplus B$, where A and B are rings in which B* is commutative. *Then R has a (changeless) involution if and only if G has an involution.*

Proof : One way is clear from Lemma 2.3-(b). Assume that A has an involution $*_A$. Define $*_R$ on R by

$$r^{*_R} = (a^{*_A}, b)$$

Then, $*_R$ is a unary operation on R and for $r_1 = (a_1, b_1), r_2 = (a_2, b_2) \in R$,

$$(r_1 r_2)^{*_R} = ((a_1 a_2)^{*_A}, b_1 b_2) = ((a_2^{*_A} a_1^{*_A}), b_2 b_1) = r_2^{*_R} r_1^{*_R}.$$

The rest is as in Lemma 2.3-(c). ■

3. Cyclic Groups with Involution

Let G be an infinite cyclic group. Following [8], there are two involutions on G, the identity involution and the involution $a^* = -a$; of taking inverse. If G is a finite cyclic group of order n, then $Aut(G)$ consists of all automorphisms, α_k : $a \rightarrow ka$, where $1 \le k \le n$ and $(k, n) = 1$. Moreover,

$$Aut(G) \cong U\left(\mathbb{Z}_n\right)$$

(the multiplicative group of units of the ring $\mathbb{Z}_n$). Since

$$\alpha_{n-1} : a \rightarrow (n-1)\, a$$

is the only automorphism of order 2, $Aut(G)$ has only two automorphisms of order two; the identity mapping and α_{n-1} (of taking inverse), so G has only two involutions.

From this introduction,we note that every cyclic group has two involutions; namely the identity mapping and the mapping of taking inverse. Moreover, every subgroup of a cyclic group is closed under these involutions.

Proposition 3.1. *Let G be an additive abelian group, $G = H \oplus K$, and let H and K be cyclic subgroups of G. If $(|H|, |K|) \neq 1$, then*

(a) *G has exactly four involutions, namely:*

$$g^* = (h, k),\ g^* = (-h, k),\ g^* = (h, -k),\ g^* = (-h, -k) \text{ and } g^* = (h, k).$$

(b) *Every subgroup of G is closed under involution.*

Proof : (a) By Lemma 2.3, H and K are *-subgroups. Since H and K are cyclic, H and K,each, has two involutions; the identity involution and $* : a \to -a$. Hence again by Lemma 2.3, G has exactly the given four involutions.

(b) By Theorem 8.1 in [4], any subgroup H of G is a direct sum of two cyclic subgroups, or it is cyclic. Hence by (a), H is a *-subgroup. ■

The following immediate result gives the number of involutions of abelian groups.

Corollary 3.2. *Let G be an additive abelian group. If $G = \bigoplus_{i=1}^{n} H_i$, where each H_i is a cyclic subgroup of G such that $(|H_i|, |H_j|) \neq 1,\ 1 \leq i, j \leq n$, then G has 2^n involutions.*

Proposition 3.3. *Let R be a ring with involution such that $R^+ = G$. Then R has only the identity involution in case any one of the following holds:*
(1) *G is a cyclic group.*
(2) *G is a direct sum of cyclic subgroups.*

Proof : (1) Let G be cyclic. Since R is an involution ring, G has either the identity involution or the involution $* : a \to -a$. However, $-(ab) \neq (-b)(-a)$, for all $a, b \in R$. Hence R has the identity involution only.
(2) If $G = H \oplus K$, and H and K are cyclic subgroups of G, then by Proposition 3.1, G has four involutions. But then again by (1) , G has only one involution. ■

Definition 3.4. By *a *-cyclic group H* , we mean a *-group generated by one element.
This means that,

$$H = (a)^* = (a, a^*) = (a) + (a^*).$$

Let G be a cyclic group, then $G = (a) = (a^*)$ and $G = (a) + (a^*)$, so G is a *-cyclic group. The converse of this fact is not always true.
For example in the group

$$(M_{2\times 2}(\mathbb{Z}_3), +)$$

with the transposed involution, let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whence $a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Obviously, the *-cyclic group $H = (a) + (a^*)$ is not a cyclic group.

Proposition 3.5. *Let $G = H \oplus K$. If H and K are *-cyclic groups such that $H = (a)^*$, $K = (b)^*$, $(|a|, |b|) = 1$. Then G is *-cyclic.*

Proof : The given condition

$$(|a|, |b|) = 1$$

implies that $(a) \oplus (b)$ is a cyclic group generated by (a, b) and $(a^*) \oplus (b^*)$ is a cyclic group generated by (a^*, b^*). But,

$$G = H \oplus K = (a) + (a^*) \oplus (b) + (b^*) = (a) \oplus (b) + (a^*) \oplus (b^*).$$

Hence G is *-cyclic with $G = ((a, b))^*$.■

Proposition 3.6. *If G is a *-cyclic group, then any *-subgroup of G is a *-cyclic subgroup.*

Proof : A *-cyclic group is either torsion or torsion free.First assume that G is torsion free and let

$$G = (a)^* = (a, a^*).$$

If

$$(a) \cap (a^*) \neq 0,$$

then $na^* = ma \neq 0$, for some integers m and n. This implies $na = ma^*$. So, $na - na^* = ma^* - ma$, from which $n(a - a^*) = m(a^* - a) = -m(a - a^*)$ and so $(n + m)(a - a^*) = 0$. Since G is torsion free, $a - a^* = 0$ implies $a = a^*$,whence $(a) \cap (a^*) = 0$ and $G = (a) \oplus (a^*)$.

Secondly assume that G is torsion, $G = (a) + (a^*)$, and $|a| = |a^*| = k$.Let $g \in G$, $g = ma + na^*$, for some integers m, n. Since $k(ma + na^*) = 0$,it follows that $|g| \leq k$, and a, a^*have maximal orders. Hence $G = (a) \oplus (a^*)$, from[6],page 81.Thus in both cases,$G = (a) \oplus (a^*)$.

If $H \leq^* G$, then $H = (b) \oplus (c)$, where $(b) \leq (a)$ and $(c) \leq (a^*)$. Then $(b) = (ma)$ and $(c) = (na^*)$,whence

$$H = (ma) \oplus (na^*).$$

Since H is *-subgroup, $ma^* + na \in H$. But $ma^* \in (na^*)$, so $m > n$ and $na \in (ma)$, so $n > m$. Therefore $n = m$ and

$$H = (na) \oplus (na^*).$$

Hence $H = (na)^*$ and H is a *-cyclic subgroup of G. ■

Proposition 3.7. *Let x_1 and x_2 be elements of a group G such that a prime $p|\ |x_1|\,,|x_2|$. If $G=(x_1)^*\oplus(x_2)^*$, then there exist y_1, $y_2\in G$ such that $(y_1)^*\leq(x_1)^*$ and $(y_2)^*\leq(x_2)^*$.*

Proof : Let

$$G=(x_1)+(x_1^*)\oplus(x_2)+(x_2^*).$$

If p is a prime such that $p\mid|x_1|$, then there exists $y_1\in(x_1)$ such that, $p|\ |y_1|$ and $|y_1|$ divides $|x_1|$. Consequently,

$$(y_1)\leq(x_1)\ \text{ and }\ (y_1^*)\leq(x_1^*).$$

Similarly there is $y_2\in(x_2)$ such that

$$(y_2)\leq(x_2)\ \text{ and }\ (y_2^*)\leq(x_2^*).$$

Hence, it is concluded that

$$(y_1)+(y_1^*)\leq(x_1)+(x_1^*)$$

and

$$(y_2)+(y_2^*)\leq(x_2)+(x_2^*),$$

that is, $(y_1)^*\leq(x_1)^*$ and $(y_2)^*\leq(x_2)^*$. ■

4. Strongly principal *-Ideal Ring Groups.

As it is mentioned before that the notion of strongly principal ideal groups for associative rings was introduced and investigated thoroughly in [2] and [3]. Motivated by these concepts, we introduce here strongly principal *-ideal ring groups for rings with involution and study their structural properties by attaching involution on their corresponding ground groups.

Definitions 4.1. Let R be a ring with involution. For some $a\in R$, one may write:

$$I=\langle a\rangle^*=\mathbb{Z}a+\mathbb{Z}a^*+aR+Ra+RaR+a^*R+Ra^*+Ra^*R.$$

Clearly I is an ideal of R closed under involution and is called the principal *-ideal generated by a.

One may deduce that

$$I=\langle a\rangle+\langle a^*\rangle=\langle a,a^*\rangle\,.$$

A ring with ivolution $*$ is a *principal *-ideal ring* if each *-ideal is a principal *-ideal. Moreover, we say that a group G is *strongly principal *-ideal ring group*, if G is not nil, and every ring R with involution satisfying $R^2 \neq 0$, and $G = R^+$, is a principal *-ideal ring.

Lemma 4.2. *Let $G = H \oplus K$, $H \neq 0$, $K \neq 0$, be a strongly principal *-ideal ring group. Then H and K are either both *-cyclic or both nil.*

Proof : Suppose that H is not nil. Let S be a *-ring with $S^+ = H$ and $S^2 \neq 0$ and let T be the zero ring on K. By Corollary 2.5, the ring direct sum $R = S \oplus T$ is a ring with involution satisfying $R^+ = G$ and $R^2 \neq 0$. Since T is a *-ideal in R, $T = \langle x \rangle^*$. Clearly $K = T^+ = (x)^*$. Therefore K is not nil. Interchanging the roles of H and K yields that H is *-cyclic. ■

Corollary 4.3. *Let $G = H \oplus K$, $H \neq 0$, $K \neq 0$, be a strongly principal *-ideal ring group. Then H and K are *-cyclic.*

Proof : It suffices to negate that H and K are both nil. Let $R = (G, \cdot)$ be a ring with involution satisfying $R^2 \neq 0$.

1) Suppose that $R^2 \subseteq K$. There exist $h_0 \in H$, $k_0 \in K$, such that $R = \langle h_0 + k_0 \rangle^*$.

Let $h \in H$, since $h \in R$, there exist integers n and m, and $x \in R^2$ such that

$$h = n(h_0 + k_0) + m(h_0 + k_0)^* + x.$$

However, $x \in K$, so

$$h = nh_0 + mh_0^*$$

and H is *-cyclic, contradicting the fact that H is nil.

2) Suppose that $R^2 \nsubseteq K$. For all $g_1, g_2 \in G$, define $g_1 \circ g_2 = \pi_H(g_1 \cdot g_2)$, where π_H is the natural projection of G onto H. Since

$$(g_1 \circ g_2)^* = (\pi_H(g_1 \cdot g_2))^* = \pi_H(g_1 \cdot g_2)^* = \pi_H(g_2^* \cdot g_1^*) = g_2^* \circ g_1^*,$$

hence $S = (G, \circ)$ is a ring with involution satisfying $S^2 \subseteq H$. The argument employed in (1) yields that K is *-cyclic which contradicts the fact that K is nil.■

Theorem 4.4. *Let $G \neq 0$ be a torsion group. If G is cyclic or $G \cong (x_1) \oplus (x_2)$, where $x_1 \neq x_2$, $|x_1| = |x_2| = p$ is a prime, then G is strongly principal *-ideal ring group.*

Proof : Assume that either G is cyclic or as given in the hypothesis. Then by Proposition. 3.3, any ring R with $R^+ = G$ has only the identity involution. By [2, Theorem 4.2.3] ,G is strongly principal *-ideal ring group.■

Theorem 4.5. *Let G be a torsion strongly principal *-ideal ring group. Then G is *-cyclic group or $G = (x_1)^* \oplus (x_2)^*$, with $|x_i| = p$, a prime,* where $i = 1, 2$.

Proof : Suppose that G is a strongly principal *-ideal ring group. Let G be indecomposable. Then by [2, Corollary 1.1.5], $G \cong \mathbb{Z}_{p^n}$, p a prime, $1 \leq n \leq \infty$. If $n = \infty$, then G is divisible by [2, Proposition.1.1.3] and so G is nil, by [2, Theorem 2.1.1], which is a contradiction. Hence G is cyclic and so *-cyclic.

Next, suppose that $G = H \oplus K$, $H \neq 0, K \neq 0$, by Lemma 4.2, either H and K are both *-cyclic or both nil. If H and K are nil, then they are both divisible, so G is nil, by [2, Theorem 2.1.1] which is again a contradiction. Therefore $G = (x_1)^* \oplus (x_2)^*$, with $|x_i| = n_i$, $i = 1, 2$. If $(n_1, n_2) = 1$, then G is *-cyclic, by Proposition. 3.5. Otherwise, let p be a prime divisor of (n_1, n_2). Then by Proposition. 3.7,

$$G = (y_1)^* \oplus (y_2)^* \oplus H$$

with

$$|y_i| = p^{m_i}, i = 1, 2, \ and \ 1 \leq m_1 \leq m_2.$$

Since $(y_1)^* \oplus (y_2)^*$ is neither * cyclic nor nil, $H = 0$ by Lemma 4.2. The products

$$y_i . y_j = p^{m_2 - -1} y_2, y_i . y_j^* = 0 \ where \ i, j = 1, 2,$$

induce a *-ring structure R on G with $R^2 \neq 0$. Therefore, $R = \langle s_1 y_1 + s_2 y_2 \rangle^*$, s_1 and $s_2$are integers. Every element $x \in R$ has the form:

$$x = k_x s_1 y_1 + (k_x s_2 + m_x p^{m_2 - 1}) y_2 + k'_x s_1 y_1^* + (k'_x s_2 + m'_x p^{m_2 - 1}) y_2^*$$

where k_x, m_x, k'_x, and m'_x are integers. In particular,

$$y_1 = k_{y_1} s_1 y_1, \ and \ y_2 = (k_{y_2} s_2 + m_{y_2} p^{m_2 - 1}) y_2.$$

Hence if $m_2 > 1$, then

$$k_{y_1} s_1 \equiv 1 (mod p), \ and \ k_{y_2} s_2 + m_{y_2} p^{m_2 - 1} \equiv 1 (mod \ p),$$

which imply that $p \nmid k_{y_1}$and $p \nmid s_2$. However

$$k_{y_1} s_2 + m_{y_1} p^{m_2 - 1} \equiv 0 (mod \ p).$$

So either $p | k_{y_1} \ or \ p | s_2$ which is a contradiction. Therefore $m_1 = m_2 = 1$.■

Theorem 4.6. *Let G be a torsion group which is either *- cyclic or $G \cong (x_1)^* \oplus (x_2)^*$ with $|x_i| = p$,a prime,where $i = 1, 2$. Then for any *-ring R with $R^+ = G$, R is a principal *-ideal ring.*

Proof : By Proposition. 3.6, non trivial *-cyclic groups are clearly principal *-ideal ring groups. Let

$$G = (x_1)^* \oplus (x_2)^*$$

with

$$|x_i| = p, i = 1, 2$$

and let R be a *-ring with $R^+ = G$ and $R^2 \neq 0$. If I is a proper *-ideal in R, then $|I| = 1,\ p$, or p^2, and so I is a *- ideal generated by one element, we may assume that $R \neq < x_i >^*$, $i = 1, 2$. Hence

$$< x_i >^{*+} = (x_j)^*$$

for $i = 1, 2$. This implies the following three relations:

$$x_i x_j = k_i x_i, \quad 0 \leq k_i < p, \text{ if } i = j, \ i = 1, 2, \text{ either } k_1 \neq 0 \text{ or } k_2 \neq 0,$$

$$x_i x_j = 0, \text{ if } i \neq j, \quad i, j = 1, 2,$$

and

$$x_i^* x_j = 0, \text{ for all } i, j = 1, 2.$$

Put

$$I = \langle x_1 + x_2 \rangle^* .$$

Suppose that $k_1 \neq 0$. Let r, s be integers such that $rk_1 + sp = 1$. Then

$$r x_1 (x_1 + x_2) = r k_1 x_1 = (1 - sp) x_1 = x_1$$

and

$$r (x_1 + x_2)^* x_1^* = r k_1 x_1^* = (1 - sp) x_1^* = x_1^*.$$

Hence $x_1 \in I$ and $x_1^* \in I$, so

$$(x_1 + x_2) - x_1 = x_2 \in I$$

and

$$(x_1 + x_2)^* - x_1^* = x_2^* \in I.$$

Therefore $I = R$. If $k_2 \neq 0$, then the above argument, reversing the roles of the indices $1, 2$ again yields $I = R$. ■

Theorem 4.7. *There are no mixed strongly principal *-ideal ring groups.*

Proof : Let G be a mixed strongly principal *-ideal ring group. G is decomposable by [2, Corollary.1.1.5], so by Lemma 4.2, $G = H \oplus K$, $H \neq 0, K \neq 0$, with H and K both *-cyclic, or both nil.

1) Suppose that H and K are both nil. There are no mixed nil groups by [2, Theorem 2.1.1]. So, we may assume that H is a torsion group, and that K is torsion free. Let R be a *-ring with $R^+ = G$ and $R^2 \neq 0$. Clearly H is a *-ideal in R and so $H = \langle h \rangle^*$. Let $|h| = m$, then $mH = 0$. By [2, Theorem 2.1.1], H is divisible, and therefore not bounded, a contradiction.

2) Suppose that $H = (x)^*$ and $K = (e)^*$ with $|x| = n$, and $|e| = \infty$. The products

$$x^2 = xe = ex = e^*x = ex^* = xe^* = x^*e = 0 \text{ and } e^2 = ne$$

induce a *-ring structure R on G satisfying $R^2 \neq 0$. Therefore there exist integers s and t such that $R = \langle sx + te \rangle^*$. Every $y \in R$ is of the form

$$y = m_y sx + (m_y + u_y n)te + m'_y sx^* + (m'_y + u'_y n)te^*,$$

with m_y, m'_y, $u_y$and u'_y integers. In particular, $(m_e + u_e n)t = 1$. Hence $t = \pm 1$. Therefore, $m_x + u_x n = 0$ and so $n | m_x$. However, $x = m_x sx = 0$, is a contradiction. ■

Theorem 4.8. *Let G be a torsion free strongly principal *-ideal ring group. Then G is either indecomposable, or is the direct sum of two nil groups.*

Proof : By Lemma 4.2. it suffices to negate that

$$G = (x_1)^* \oplus (x_2)^*, \ x_i \neq 0, \ i = 1, 2.$$

Suppose this is so, the products:

$$x_i x_j = 3x_i \text{ and } x_i^* x_j \ = 0 \text{ for } \ i = j = 1, 2,$$

$$x_i x_j = x_i^* x_j = 0, \text{ for } i \neq j$$

induce a ring structure R on G with involution * satisfying $R^2 \neq 0$. Therefore there exist nonzero integers k_1, k_2 such that

$$R = \langle k_1 x_1 + k_2 x_2 \rangle^* .$$

Every $x \in R$ is of the form:

$$x = (r_x + 3s_x)k_1x_1 + (r_x + 3t_x)k_2x_2 + (r'_x + 3s'_x)k_1x_1^* + (r'_x + 3t'_x)k_2x_2^*$$

where $r_x,\ r'_x,\ s_x,\ s'_x,\ t_x,\ t'_x$ are integers.From

$$r_{x_1} + 3s_{x_1} = \pm 1,$$

it follows that

$$r_{x_1} \equiv \pm 1 (mod\ 3).$$

However,

$$r_{x_1} + 3t_{x_1} = 0$$

implies

$$r_{x_1} \equiv 0 (mod\ 3),$$

which is a contradiction.■

Lemma 4.9. *Let G and H be torsion free groups with $G \overset{*}{\cong} H$. Then G is a strongly principal *-ideal ring group if and only if H is.*

Proof : Let $f : H \longrightarrow G$ be a *-isomorphism such that G is a strongly principal *-ideal ring group. Let $R = (H, \cdot)$ be a ring with involution $*$ such that $R^2 \neq 0$. The product

$$g_1 \circ g_2 = f(h_1 \cdot h_2),$$

where, $g_1 = f(h_1)$ and $g_2 = f(h_2)$, for all $g_1, g_2 \in G$, induces a ring structure $S = (G, \circ)$ with $S^2 \neq 0$. Then G is a group with involution ∇. Since

$$(g_1 \circ g_2)^{\nabla} = [f(h_1 \cdot h_2)]^{\nabla} = f(h_1 \cdot h_2)^* = f(h_2^* \cdot h_1^*) = g_2^{\nabla} \circ g_1^{\nabla}.$$

Hence S has an involution. Let $I \lhd^* R$, then $f(I) \lhd^{\nabla} S$ and there exists $g \in G$ such that $f(I) = \langle g \rangle^{\nabla}$, with $g = f(h)$, $h \in H$. We claim that $I = \langle h \rangle^*$. Clearly $\langle h \rangle^* \subseteq I$. Let $x \in I$, then $f(x) \in \langle g \rangle^{\nabla}$ and so

$$f(x) = ng^{\nabla} + mg + g \circ y_1 + g^{\nabla} \circ y_2 + z_1 \circ g + z_2 \circ g^{\nabla}$$

where m, n are integers and $y_1 = f(\ h'_1)$,$y_2 = f\ (h'_2)$, $z_1 = f\ (h'_1)$ and $z_2 = f\ (h'_2) \in G$ .Thus

$$\begin{aligned} f\ (x) &= nf\ (h^*) + mf\ (h) + f\ (h \cdot h'_1) + f\ (h^* \cdot h'_2)\ + f\ (h'_1 \cdot h)\ + f\ (h'_2 \cdot h^*) \\ &= f\ (nh^* + mh\ + h \cdot h'_1 + h^* \cdot h'_2 + h'_1 \cdot h\ + h'_2 \cdot h^*) \end{aligned}$$

which concludes that $x \in \langle h \rangle^*$. Hence $I = \langle h \rangle^*$. ■

Theorem 4.10: *Let G be a torsion group. Then the following are equivalent:*

(1) *G is bounded*

(2) *G is a principal *-ideal ring group.*

Proof : (1) ⇒ (2): Suppose that $nG = 0$ and n is a positive integer. Then

$$G = \underset{p|n}{\oplus} \left[\underset{\alpha_k}{\oplus} \mathbb{Z}_{p^k} \right]$$

where p is a prime with $p^k \mid n$ and α_k a cardinal number ,by [2, Proposition 1.1.9]. For each $p^k \mid n$, put

$$H_{p^k} = \underset{\alpha_k}{\oplus} \mathbb{Z}_{p^k}.$$

Then

$$G = \underset{p^k|n}{\oplus} H_{p^k},$$

and there exists a commutative principal ideal ring R_{p^k} with unity and

$$R^+_{p^k} = H_{p^k}$$

for all $p^k \mid n$,by [5, Lemma 122.3]. The ring direct sum

$$R = \underset{p^k|n}{\oplus} R_{p^k}$$

is a principal *-ideal ring with the identity involution satisfying $R^+ = G$ and $R^2 \neq 0$.

(2) ⇒ (1): Let R be a principal *-ideal ring with $R^+ = G$. Then $R = \langle x \rangle^*$ and $n = |x|$. So $nG = 0$. ■

Theorem 4.11: *Let G be a mixed group. Then*

(1) *If G is a principal *-ideal ring group, then G_t is bounded and G/G_t is a principal *-ideal ring group.*

(2) *Conversely, if G_t is bounded and if there exists a unital principal *-ideal ring with additive group G/G_t, then G is a principal *- ideal ring group.*

Proof : (1) Let R be a principal *-ideal ring with $R^+ = G$. Since G_t is a *-ideal in R, $G_t = \langle x \rangle^*$ and $nG_t = 0$, $n = |x|$. Now $G = G_t \oplus H$ and $H \cong G/G_t$ by [2, Proposition. 1.1.2]. Now

$$R = \langle a + y \rangle^* , a \in G_t, 0 \neq y \in H.$$

Suppose that $R^2 \subseteq G_t$ and let $h \in H$. Then there exist integers $k_n,\ k'_n$ such that,

$$h = k_n y + k'_n y^* + b,$$

with $b \in R^2$. Since $R^2 \subseteq G_t,\ b = 0$, and

$$h = k_n y + k'_n y^*.$$

Therefore $H = (y)^*$. By Proposition 3.6, H is a principal *- ideal ring group.

If $R^2 \nsubseteq G_t$, then $\bar{R} = R/G_t$ is a principal *-ideal ring with $\bar{R}^+ \cong G/G_t$, and $\bar{R}^2 \neq 0$.

(2) Conversely, suppose that G_t is bounded, and that there exists a unital principal *-ideal ring T with $T^+ = G/G_t$. Hence

$$G \cong G_t \oplus G/G_t,$$

by [2, Proposition. 1.1.2]. There exists a principal *-ideal ring S with unity and * is the identity involution such that $S^+ = G_t$, from [5, Lemma 122.3]. Let

$$R = S \oplus T$$

with e, f the unities of S and T, respectively. Then R is a ring with involution *, by Corollary 2.5. Let I be a *-ideal in R, then

$$I = (I \cap S) \oplus (I \cap T).$$

Now, $I \cap S \lhd^* S$ and so

$$I \cap S = \langle x \rangle^* .$$

Similarly

$$I \cap T =< y >^* .$$

Clearly,

$$\langle x + y \rangle^* \subseteq I.$$

However,

$$x = e(x + y) \in \langle x + y \rangle^* ,\ x^* = e(x + y)^* \in \langle x + y \rangle^*$$

and

$$y = f(x + y) \in \langle x + y \rangle^* , y^* = f(x + y)^* \in \langle x + y \rangle^* .$$

Hence we conclude that $I =< x + y >^*$.■

References

[1] U.A. Aburawash, On involution rings,*East-West J.Math* ,Vol. 2, No. 2, (2000), 102-126.

[2] S. Feigelstock, *Additive Groups of Rings,* Pitman (APP), 1983.

[3] S. Feigelstock, Z. Schlussel, Principal ideal and noetherian groups, *Pac. J. Math*, 75 (1978), 85-87.

[4] L. Fuchs, *Infinite Abelian Groups, Vol. I,* Academic Press, New York, 1970.

[5] L. Fuchs, *Infinite Abelian Groups, Vol. II,* Academic Press, New York, 1973.

[6] T.W. Hungerford, *Algebra*, Springer Science+Business Media, 1974.

[7] N.V. Loi, On the structure of semiprime involution rings, *Contr. to General Algebra, Proc. Krems Cons., North-Holland* (1990), 153-161.

[8] D.J.S. Robinson, *A Course in the Theory of Groups,* Springer, New York, 1991.

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